# Noise-Induced Linearisation and Delinearisation

M I Dykman

Department of Physics, Stanford University,

Stanford, California 94305, USA.

D G Luchinsky[1]

School of Physics and Materials, Lancaster University,

Lancaster, LA1 4YB, UK.

R Mannella

Dipartimento di Fisica, Università di Pisa,

Piazza Torricelli 2, 56100 Pisa, Italy.

P V E McClintock, H E Short, N D Stein and N G Stocks[2]

School of Physics and Materials, Lancaster University,

Lancaster, LA1 4YB, UK.

Abstract

It is demonstrated, by means of analogue electronic simulation and theoretically, that external noise can markedly change the character of the response of a nonlinear system to a low-frequency periodic field. In general, noise of sufficient intensity *linearises* the response. For certain parameter ranges in particular cases, however, an increase in the noise intensity can sometime have the opposite effect and is shown to *delinearise* the response. The physical origins of these contrary behaviours are discussed.

[1]Permanent address: All Russian Institute for Metrological Service, Andreevskaya nab 2, 117965 Moscow, Russia.

[2]Present address: Department of Engineering, University of Warwick, Coventry, CV4 7AL, UK.

INTRODUCTION

Noise in physical systems is Janus-like, with two oppositely directed faces. Both of them are relevant to an understanding of complex systems[1]. The negative, destructive, and therefore ugly, face of noise - which is probably the more familiar to most physicists - corresponds to the familiar blurring by random fluctuations of otherwise well-defined quantities, the randomization of initially ordered systems, and the destruction of fine detail in intricate patterns. The universality of noise in macroscopic physical systems requires that this aspect be taken explicitly into account in any measurement, because it gives rise to a "random error". The ugly face of noise is especially obtrusive in relation to studies of chaotic phenomena[2] in real systems, where it is usually the effect of internal or external noise that sets the practical lower limit on the range of co-ordinate scales over which, for example, fractal effects can persist.

It is perhaps less well known, but one of the major themes of the present volume, that noise can also exhibit a face that is beautiful, in the sense that its effects can also be positive and creative. In stochastic resonance[3], for example, noise can enhance a weak periodic signal in a nonlinear system. It can give rise to noise-induced transitions[4] in which a discontinuous change occurs in the number of extrema in a probability density. Noise can be used to overcome[5] the phase-locking of ring-laser gyroscopes caused by nonidealities of their mirrors, thereby linearizing the response. It can create spatial structures in liquid crystals[6] and stabilise them in convecting fluids[7]. Noise also appears to play an important role in the maintenance of consciousness[8]. These and other examples are discussed in earlier chapters of this volume and in the pages that follow.

It is with the positive, creative, aspect of noise that the present chapter is mainly concerned. We consider the effect of external noise on the passage of a periodic signal (in principle, of any form) through a nonlinear system (in principle, of any form). For a signal of finite amplitude, on account of the nonlinearity, the output signal $q(t)$ in the absence of noise will naturally be distorted compared to the input. The experimental observation that we wish to report and discuss is that this distortion can apparently always be removed by the addition at the input of external white[3] noise of sufficient intensity:

[3]For convenience, we will concentrate on the effects of white or quasi-white noise. Most of the discussion, however, is equally applicable qualitatively, if not always quantitatively, to the case of non-white noise, provided that its correlation time is shorter than the characteristic times of the system under

a process that might reasonably be described by the term *noise-induced linearisation*. The linearized output resulting from this procedure is inevitably noisy and so, to focus attention on what happens to the periodic signal itself, in what follows we will discuss how the ensemble average $\langle q(t) \rangle$ of the output varies with relevant parameters, for example with the noise intensity at the input.

These ideas are illustrated by the experimental data shown in Figures 1 and 2, obtained from an analogue electronic circuit model of an overdamped bistable system (see below). The input to the model in each case is the waveform shown at the top, whose frequency is low compared to the system's reciprocal relaxation time. For negligible noise intensity $D$, the output response of the system is grossly distorted, as shown by the upper $\langle q(t) \rangle$ traces. As the noise intensity is increased, however, the distortion steadily diminishes in each case until, in the lowest trace, the output can be seen to be a faithful reproduction of the input waveform. With some exceptions, which will be discussed, this scenario has been found experimentally to hold for a very wide range of nonlinear systems - monostable as well as bistable, underdamped as well as overdamped, chaotic as well as regular - and shapes of the input waveform.

It must be emphasized that the investigations are still in progress. Our intention here is to identify noise-induced linearisation as an interesting and potentially important fluctuational phenomenon that does not seem to have received attention previously in its own right, and whose range of occurrence appears to be very wide. We make no claim to a complete understanding of it at this stage, however, and would point out that there are still a number of open questions remaining to be resolved.

The plan of the chapter is this. There follows a general discussion of the physical basis of the dramatic results of Figures 1 and 2. It is also pointed out that the obverse phenomenon of noise-induced *delinearisation* can occur under certain special circumstances. The third section presents a discussion of noise-induced linearisation in relation to the well-known and much-studied case of overdamped motion in a bistable potential; but it is argued that, within a certain parameter range, the delinearisation phenomenon is also to be anticipated in the same system. Then there is a section describing investigations of these phenomena in a quite different type of system - an underdamped monostable

consideration.

nonlinear oscillator - for which noise-induced delinearisation, followed by linearisation at higher noise intensities, has been both predicted and observed. Finally, the results are summarized and general conclusions are drawn.

## PHYSICAL BASIS OF NOISE-INDUCED LINEARISATION AND DELINEARISATION

The claim for novelty in the results of Figures 1 and 2, and in the discussion that follows must be carefully qualified, because the basic idea of linearisation by added noise will already be familiar to some through specific applications in certain particular fields of science and engineering. However the proposition that noise-induced linearisation should exist *as a general phenomenon* has not, to our knowledge, been enunciated previously. Nor has it been tested experimentally, as we describe below, for a variety of physical systems.

We note that the word *linearisation* is commonly used in two rather different senses, and that these are exemplified by the results of Figures 1 and 2. The fact that a sinusoidal input can pass through the system without significant change of shape, as occurs for strong noise in the lowest trace in Figure 1, implies linearity in the sense of a direct proportionality between the amplitudes of output and input; this need not necessarily, however, imply that the constant of proportionality must be frequency-independent. On the other hand, the results of Figure 2, for a sawtooth waveform containing not only the fundamental frequency but also its higher harmonics, imply the occurrence of linearisation in the "Hi-Fi" sense that the system becomes non-dispersive within a certain frequency range when the noise intensity is large enough. It would appear that, for the types of system considered,[4] linearisation in this latter sense automatically implies linearisation in terms of the amplitude as well, but not the converse.

The physical origin of both forms of nonlinearity can readily be understood, at least qualitatively, in the following terms. Where the amplitude response of a system to a periodic force is nonlinear, this arises because the amplitude of the vibrations induced by

[4]Obviously, one could try to devise special circumstances which might violate this rule. One possible example would be a logarithmic amplifier. However, even though this would be dispersion-free but highly nonlinear for low noise intensities, it seems likely nonetheless that the response would be linearized by noise, just like the other cases that we consider.

the force is large compared to some characteristic length scale of the system. The scale in question is determined by the structure of the region of phase space being visited by the system and by corresponding features in the dynamics. The effect of noise is to smear the system over a larger region of phase space, so that a variety of different scales and frequencies then become involved in the motion, even in the absence of periodic driving, and the effective characteristic scale will usually increase as a result. For sufficiently large noise intensities, therefore, the amplitude of the force-induced vibrations will become small compared to the scale (eg small compared to the average size of the noise-induced fluctuations), so that the nonlinearity in the response amplitude is correspondingly reduced. Because the system is then spending an increasing proportion of its time far away from its attractor(s), at coordinate values where the characteristic time in responding to an additional perturbation (the periodic force) will in general be quite different and often, in practice, must shorter, than that near the attractor, there will be one or more ranges of frequency for which dispersion is likely to decrease. Although the two forms of linearisation arise, ultimately, through the same physical processes - the effect of noise in smearing the system over a larger region of its phase space - there is no reason to expect that their onsets will occur at the same noise intensity.

All peaks in spectral densities of fluctuations are broadened by an increase in noise intensity, both for overdamped and underdamped systems. Where the suppression of dispersion arises (see below) through the broadening of a peak centered on zero frequency, one would expect the process to require a considerably stronger noise intensity than that necessary to give linearisation in terms of the amplitude of a sinusoidal waveform at the fundamental frequency $\Omega$. This is partly because of the need to linearise the response up to larger frequencies, in the case of a more complicated waveform of fundamental frequency $\Omega$, in order to be able to accommodate the higher Fourier components at multiples of $\Omega$, and partly because of the need to obtain a frequency-independent susceptibility once linear response has been achieved.

When the noise intensity is sufficient to provide a linear response in terms of amplitudes, the situation can conveniently be discussed in terms of classical linear response theory[9]. Quite generally, the ensemble-averaged response of the sytem to a periodic force of frequency $\Omega/2\pi$ can be written in the form

$$\langle q(t)\rangle = \sum_n a(n)[\cos n\Omega t + \phi(n)] \tag{1}$$

We ignore here any transient phenomena associated with the initial application of the stochastic or periodic forces, and suppose that the system has already reached a stationary state. We are assuming that the noise intensity is large enough (for the given amplitude and frequency of periodic force) that the response can be characterised by a linear susceptibility. The response to a cosine driving force, $A\cos\Omega t$, can then be well approximated by

$$\langle q(t)\rangle = a\cos(\Omega t + \phi) \tag{2}$$

because the higher harmonics in Eq.(1) are negligible, with

$$a \equiv a(1) = A|\chi(\Omega)| \tag{3}$$

$$\phi \equiv \phi(1) = -\arctan[\mathrm{Im}\chi(\Omega)/\mathrm{Re}\chi(\Omega)] \tag{4}$$

The susceptibility $\chi(\Omega)$ can be obtained[10] from the fluctuation dissipation theorem[9].

$$\mathrm{Re}\chi(\Omega) = \frac{2}{D}\mathrm{P}\int_0^\infty d\omega_1[\omega_1^2/(\omega_1^2-\Omega^2)]Q_0(\omega_1) \tag{5}$$

$$\mathrm{Im}\chi(\Omega) = (\pi\Omega/D)Q_0(\Omega) \tag{6}$$

Here, $D$ is the noise intensity, P implies the Cauchy principal part and $Q_0(\omega)$ represents the spectral density of fluctuations of the system in the absence of the periodic force. The response to more complicated periodic forces can be considered in a similar way. Because of the frequency dependence of $\chi(\Omega)$, the relative amplitudes of Fourier components at the harmonics will be modified by the system, so that the response $\langle q(t)\rangle$ will in general be distorted as compared to the input, as observed for weak and intermediate noise intensities in Figures 1 and 2.

It is also evident from Eqs.(5), (6), however, that where there is a range (or are ranges) of frequency $\omega$ for which $Q_0(\omega)$ is relatively flat, the frequency dependences of $\chi(\Omega)$ and therefore of $a, \phi$ will also be correspondingly weak. Consequently, noise induced

linearisation in the sense of removing dispersion is to be anticipated whenever an increase of $D$ has the effect of flattening $Q_0(\omega)$ within the frequency range of interest.

It is clear from the foregoing discussion that there can be no generally applicable quantitative theory of noise-induced linearisation. The behaviour seen in practice will naturally depend on the system under study and, in particular, on the way in which its $Q_0(\omega)$ evolves with noise intensity. However, the quite general broadening effect exerted by noise on the peaks of $Q_0(\omega)$ implies that there will usually be at least one range of frequencies (often centred at zero frequency) for which not only linearisation in terms of the amplitudes, but also a reduction of dispersion is to be anticipated.

It is important to note, however, that exactly the same physical processes which give rise to noise-induced linearisation can also, under special circumstances, give rise to the opposite effect of noise-induced delinearisation. The latter phenomenon is to be anticipated if the additional frequencies that become involved (as the result of an increase in noise intensity) resonate with the periodic force or with one of its harmonics, or if the noise modifies the characteristic reciprocal relaxation time of the system so that it corresponds to the frequency of the periodic force. The standard form of stochastic resonance is related to the latter condition. In such cases, it is to be expected that the promotion of nonlinearity by noise at intermediate intensities will be followed by the more general phenomenon of noise-induced linearisation at still higher noise intensities owing to the usual noise-induced broadening of the relevant spectral peak(s).

We now consider, in the following sections two specific examples of systems whose response can be either linearized or delinearised by adding external noise at the input.

## NOISE-INDUCED LINEARISATION IN AN OVERDAMPED BISTABLE SYSTEM

Noise-induced linearisation in terms of the amplitude is already well known for the case of overdamped motion in a symmetrical double well potential, described by

$$\dot{q} + U'(q) = A\cos\Omega t + f(t) \tag{7}$$

$$U(q) = -\frac{1}{2}q^2 + \frac{1}{4}q^4 \tag{8}$$

where $A\cos\Omega t$ is the periodic input force and $f(t)$ is quasi-white zero-mean Gaussian noise of intensity $D$,

$$\langle f(t)f(t')\rangle = 2D\delta(t-t') \tag{9}$$

The system described by Eqs.(7)-(9) has been studied intenstively, in connection particularly with stochastic resonance[3], and its properties are to a large extent well known and understood. In particular, the zero-frequency peak in the spectral density of fluctuations $Q_0(\omega)$, caused by interwell transitions, is known to broaden rapidly with increasing noise intensity. Within the range of linear response, and for $D \ll \Delta U$ where $\Delta U$ is the depth of each potential well, which $= \frac{1}{4}$ for the model of Eq (8), the susceptibility can[11,12] be written

$$\chi(\omega) = \sum_n w_n(U''_n - i\omega)^{-1} + \frac{w_1 w_2}{D}(q_1-q_2)^2 W(W-i\omega)^{-1} \tag{10}$$

where $w_1 = w_2 = \frac{1}{2}$ are the populations of the two potential wells, the $U''_n$ are the curvatures of the bottoms of the wells at $q_n = (-1)^n$ for $n = 1$, 2 and $W = W_{12} + W_{21}$ is the sum of the interwell transition probabilities $W_{nm}$ from $n \to m$. It is immediately evident by inspection of Eq (10) that, for $\omega \ll U''_n$, there will be strong dispersion for weak noise where $W \lesssim \omega$ but negligible dispersion for stronger noise when the exponential rise of $W_{nm}$ with $D$ implies $W \gg \omega$. Figure 3(a)-(c) shows the variation of the response with frequency, for three different noise intensities. The full curves correspond to $|\chi^{(1)}|^2$; they exhibit a strong dependence on frequency for weak noise intensity (c), but become much flatter as the noise intensity is increased to a large value (a). The dashed curves indicate the squared ratio of the response at the third harmonic to that at the fundamental: they provide a quantitative measure of the dispersion, which can be seen to decrease rapidly with increasing noise intensity. Noise-induced linearisation (in the frequency sense) is therefore indeed to be anticipated for the system Eqs.(7)-(9), thereby accounting for the phenomena observed in Figures 1 and 2 which were obtained from an analogue electronic circuit[12] of conventional[13,14] design built to model this system.

Before going on to consider delinearisation/linearisation phenomena in a different type of system, we would point out that there is a parameter range within which the model described by Eqs.(7)-(9) would also be expected to display noise-induced delinearisation.

To see how this comes about, note that its response is already known to display a giant nonlinearity[15] within the parameter range where the noise has "tuned" the reciprocal interwell transition probability in such a way that the system makes on average almost exactly one pair of transitions, at almost the same phases each time, per cycle of the periodic force. Clearly, for very much lower noise intensities, such that the interwell transitions are suppressed, the response to *weak* periodic forcing about either one of the potential wells will be at least approximately linear. Consequently, there must be a parameter range (low frequency and small amplitude of the periodic force, very weak noise) for which the system exhibits some degree of noise-induced delinearisation; but, of course, it will linearize again for larger values of $D$ as already discussed.

## NOISE-INDUCED DELINEARISATION IN AN UNDERDAMPED MONOSTABLE SYSTEM

To demonstrate that noise-induced linearisation/delinearisation phenomena are in no way confined to overdamped bistable systems like the one described by Eqs.(7)-(9), we now show that similar effects are to be seen in an underdamped, monostable oscillator. This is an example of a case where delinearisation arises because the effect of the noise is to modify the oscillator's eigenfrequency in such a way as to "tune" it through a harmonic of the periodic force. The system we consider is described by the equations

$$\ddot{q} + 2\Gamma\dot{q} + U'(q) = A\cos\Omega t + f(t) \tag{11}$$

$$U(q) = \frac{1}{2}\omega_0^2 q^2 + \frac{1}{3}\beta q^3 + \frac{1}{4}\gamma q^4 \tag{12}$$

$$\langle f(t)f(t')\rangle = 4\Gamma T\delta(t-t') \tag{13}$$

where $f(t)$ is zero-mean Gaussian white noise of intensity $T$ and the damping constant is assumed small, $\Gamma \ll \omega_0$. Provided that $\gamma > 0$ and $|\beta| < 2\omega_0\gamma^{\frac{1}{2}}$, the potential of Eq (12) has a single minimum and the oscillator is monostable. Because of the anharmonic terms in Eq (12), the frequency $\omega(E)$ of the eigenvibrations depends on the oscillator energy $E = \frac{1}{2}p^2 + U(q)$, where $p = \dot{q}$ is the momentum of the oscillator. For large $E$, the frequency increases with energy, as $E^{\frac{1}{4}}$. For small energies, on the other hand,

$$\omega(E) \simeq \omega_0 + \omega_0' E$$

where

$$\omega_0' \equiv \left(\frac{d\omega(E)}{dE}\right)_{E=0} = \frac{3}{4}\gamma\omega_0^{-3} - \frac{5}{6}\beta^2\omega_0^{-5}$$

Consequently, for $(\beta^2/\gamma\omega_0^2) > 9/10$, the function $\omega(E)$ is nonmonotonic: it decreases with $E$ for small $E$ and then increases again for larger $E$: $\omega(E)$ is plotted for three values of $\beta$ in Figure 4, with $\omega_0 = \gamma = 1$. Such nonmonotonicity of $\omega(E)$ in an underdamped oscillator gives rise to a number of interesting features[16–18] in its spectral density of fluctuations that have been reported and discussed elsewhere.

A situation of particular physical interest arises when the periodic driving frequency is close to half the eigenfrequency of small amplitude vibrations,

$$|2\Omega - \omega_0| \ll \omega_0 \tag{14}$$

The absorption of a periodic force close to the subharmonic frequency $\omega_0/2$ can be strongly noise-dependent: in effect, the noise intensity can be used to change the energy of the oscillator, and hence adjust its eigenfrequency $\omega(E)$ to the second harmonic of the driving frequency, giving rise to a nearly resonant absorption. The "tuning" process is rather similar to the stochastic resonance that is seen in monostable systems[19], but it occurs here for a periodic force applied, not near the eigenfrequency, but at approximately half the eigenfrequency. The phenomenon is of considerable interest, is of some relevance to two-photon absorption processes in optics, and will be discussed in more detail elsewhere[20]. For present purposes, we are mainly concerned with the likelihood that the noise-induced resonance will correspondingly give rise to a noise-induced nonlinearity of the response.

The motion of the oscillator described by Eqs.(11)-(13) consists of random vibrations induced by the noise, varying slowly in amplitude and phase, that get mixed nonlinearly with the periodic vibrations at $\Omega$ and its overtones induced by the periodic force. To lowest order in the force amplitude, only the vibrations at frequency $\Omega$ are excited. If the noise intensity $T$ is small enough,

$$\langle q(t)\rangle^{(1)} \simeq \frac{A}{3\omega^2}\cos(\Omega t + \phi^{(1)}) \tag{15}$$

$$\phi^{(1)} \simeq -2\Gamma/3\Omega \tag{16}$$

The existence of the phase shift $\phi^{(1)}$ corresponds to a weak linear absorption of energy by the oscillator. The absorption coefficient $\kappa$ is defined as the ratio of the energy absorbed per unit time (averaged over the period) to $A^2$. In the approximation given by Eqs.(15), (16), $\kappa \simeq \kappa^{(1)}$ where

$$\kappa = A^{-2}\overline{\langle \dot{q} A \cos \Omega t \rangle} \tag{17}$$

$$\kappa^{(1)} = \Gamma/9\Omega^2 \tag{18}$$

The resultant vibrations at frequency $\Omega$ give rise in turn to vibrations at its overtones. For comparatively small $A$, and when Eq (14) holds, it is the vibrations at $2\,\Omega \simeq \omega_0$ that are of primary interest. The equation of motion for these vibrations can be obtained by making the substitution

$$q(t) \simeq \langle q^{(1)}(t) \rangle + q^{(2)}(t)$$

in Eq (11) and separating the terms oscillating at frequencies $\sim 2\Omega \simeq \omega_0$,

$$\ddot{q}^{(2)} + 2\Gamma\dot{q}^{(2)} + \omega_0^2 q^{(2)} + \beta[q^{(2)}]^2 + \gamma[q^{(2)}]^3 \simeq f(t) - \frac{8A^2}{9\omega_0^4}\beta\cos(2\Omega t + 2\phi^{(1)}) \tag{19}$$

where we have ignored the renormalisation of the frequency $\omega_0$ by $(16/9)\ \omega_0' F^2/\omega_0^2$ (which corresponds to a kind of dynamical Stark shift for a nonlinear oscillator).

Equation (11) constitutes the equation of motion of a nearly-resonantly-driven nonlinear oscillator. Its linear response to the periodic force given by the final term can be described in a very similar way to the theory of monostable stochastic resonance[19], obtaining the spectral density $Q_0(\omega)$ in the absence of the periodic force by the method developed for the tilted Duffing oscillator[16], and then using the fluctuation dissipation relations (5), (6) (with $T$ in place of $D$ for the noise intensity) to determine the susceptibility $\chi(\Omega)$. When this procedure is performed it is found that $\chi(\Omega)$, $a$ and $\phi$ are strongly dependent on noise intensity, just as anticipated on the basis of the simple physical arguments presented above in terms of stochastic "tuning" of the system.

The theoretical predictions have been tested experimentally on an analogue electronic model[13,14] of Eqs.(11)-(13); the design and operating procedures will be described in detail elsewhere[20]. The model was driven by quasi-white noise and by the periodic force $A\cos\Omega t$ with its frequency $\Omega$ chosen to be slightly less than $\omega_0/2$. With $\beta$ set so as to make $\omega(E)$ nonmonotonic (Figure 4) one would therefore anticipate a noise-induced resonance at the second harmonic for an appropriate value of $D$; one would expect in turn that the resultant absorption of energy from the periodic driving force would results in a delinearisation of what, in the absence of noise, had been a weak and, to a good approximation, linear response at $\Omega$.

For circuit parameters set to $\omega_0 = 1$, $\Gamma = 0.011$, $\beta = 1.67$, $\gamma = 1$, and with a sinusoidal drive of frequency $\Omega = 0.442$ and amplitude $A = 0.200$, the dependence on the noise intensity $T$ of the absorption coefficient $\kappa$ and phase shift $\phi$ of the response at $\Omega$ were measured and found to be as shown in Figure 5(a) and (b). It is immediately evident that $\kappa$ and $\phi$ exhibit a maximum and minimum respectively at $T \simeq 0.01$. The occurrence of noise-induced delinearisation was quantified by computation of the Fourier components of the power spectrum of the ensemble average $\langle q(t)\rangle$, measured for different noise intensities. The ratio $R$ of the intensity of the delta spike at the second harmonic to that at the fundamental frequency $\Omega$, providing a measure of the nonlinearity in terms of the amplitude, is plotted as a function of $T$ in Figure 6: it can be seen that the nonlinearity of the response rises rapidly with $T$, peaks at $T \simeq 0.01$ where the absorption and phase lag also pass through their maximum values (Figure 5), and then decreases again for larger $T$. Thus, the system (14)-(16) is exhibiting, in turn, noise-induced delinearisation and then noise-induced linearisation as $T$ is increased, just as predicted above.

## CONCLUSION

We conclude that the noise-induced linearisation of the response of a nonlinear system to a sinusoidal force is a very general phenomenon. Noise-induced linearisation in the sense that the frequency dispersion is also reduced by noise, so that a non-sinusoidal periodic signal can then pass through the system without distortion, also appears to be of wide occurrence: we suggest that it is to be anticipated for all hardening potentials provided that the fundamental frequency of the force and its relevant harmonics are very much lower then the natural frequency of small oscillations in underdamped systems, or

than the reciprocal relaxation time in the case of overdamped systems. An important *caveat*, applicable to both forms of linearisation, is that there may exist a restricted parameter range within which an increase of the noise intensity serves to delinearise the system. Whether or not a hardening potential is a necessary condition remains to be explored: it seems probably that a hardening potential is required for linearisation in underdamped systems, but possible that any type of binding potential is sufficient in the case of overdamped ones. The delinearisation phenomenon is likely to occur in cases where the frequency of the periodic force, or of one of its harmonics, falls close to an eigenfrequency (or reciprocal characteristic relaxation time) of the system.

Although a preliminary series of electronic analogue experiments on different systems has persuaded us that noise-induced linearisation is widespread, its precise range of occurrence has yet to be established. It remains possible, of course, that there are other physical mechanisms, additional to the one mentioned above, through which noise-induced delinearisation can arise. Further work, both theoretical and experimental, will be needed before these effects can be considered fully understood and characterised within the larger category of phenomena arising through the influence of noise in its positive role as a creative force.

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FIGURE CAPTIONS

1. Noise-induced linearisation for a sinewave passing through an electronic model of the overdamped double-well system given by Eqs.(7)-(9). The periodic force at the input is shown in the upper trace. The ensemble-averaged response $\langle q(t) \rangle$, measured at the output, is shown for different noise intensities $D$ in the lower traces. The amplitudes of the latter have been normalised so as to be comparable with the amplitude of the force, for easier comparison of their relative shapes.

2. Noise-induced linearisation for a sawtooth wave passing through an electronic model of the overdamped bistable system given by Eqs.(7)-(9). The periodic force at the input is shown in the upper trace. The ensemble-averaged response $\langle q(t) \rangle$, measured at the output, is shown for different noise intensities $D$ in the lower traces. The amplitudes of the latter have been normalised so as to be comparable with the amplitude of the force, for easier comparison of their relative shapes.

3. The normalised response of the system (7)-(9) to a sinusoidal driving force, as a function of its frequency $\Omega$, for three noise intensities: (a) $D = 0.5$; (b) $D = 0.3$; (c) $D = 0.02$. The full curves represent the squared ratio of the response $a(1)$ at the driving frequency to the amplitude $A$ of the driving force. The dashed curves represent the squared ratio of the response $a(3)$ at the third harmonic of the driving frequency to the response $a(1)$ at the driving frequency.

4. Plots of the eigenfrequency $\omega(E)$ of the oscillator defined by (11)-(13) as a function of the energy in the oscillations, measured relative to the bottom of the potential well, for (top to bottom) $\beta = 1.30$, 1.50 and 1.67, respectively.

5. A noise induced resonance in an electronic model of the system (11)-(13), driven at a frequency slightly less than half the eigenfrequency of small amplitude vibrations. Respectively, (a) the absorption coefficient $\kappa$ and (b) the phase shift $\phi$ between the force and the response at the fundamental are plotted at functions of noise intensity $T$.

6. Noise-induced delinearisation, followed by linearisation, in an electronic model of the system (11)-(13), operated under the same conditions as for the data of Figure

5. The ratio $R$ of the delta spike at the second harmonic to that at the fundamental in the spectral density of the ensemble-averaged response $\langle q(t) \rangle$, which provides a quantitative measure of the linearity of the system, is plotted as a function of the noise intensity $T$.